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ON THE SUMMABILITY FUNCTION AND THE ORDER FUNCTION OF DIRICHLET SERIES

BY

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§ 1. Introduction.[1]

The aim of the present paper is to give a contribution to the study of the connection between the so-called summability function $\psi(\sigma)$ and the order function $\mu(\sigma)$ of an ordinary Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$. Before stating the results of the paper we shall recall the definitions of these functions and some known theorems.

Let $f(s) = \sum a_n n^{-s}$ be an ordinary Dirichlet series which is neither everywhere divergent nor everywhere convergent. Let for every integer $r \geqq 0$ the number λ_r denote the abscissa of summability of the r^{th} order, in particular λ_0 the abscissa of convergence. Then, as shown by the author ([2], and [3], pp. 99—104),

(1) $0 \leqq \lambda_r - \lambda_{r+1} \leqq 1$ and $\lambda_r - \lambda_{r+1} \geqq \lambda_{r+1} - \lambda_{r+2}$ $(r = 0, 1, 2, \cdots)$.

When we follow M. Riesz and consider summability of *arbitrary* order $r \geqq 0$, the abscissa λ_r exists as a function of r in the interval $0 \leqq r < \infty$. In generalization of the above inequalities the function $\sigma = \lambda_r$ is a non-increasing continuous convex function with numerical slope $\leqq 1$ (see [6], pp. 57 and 60, and [8], p. 118). We introduce the number $\Omega\,(\geqq -\infty)$ as the limit $\Omega = \lim_{r \to \infty} \lambda_r$. It follows from the results just mentioned that when r increases from 0 to ∞, then λ_r will be either a strictly decreasing function which tends to Ω for $r \to \infty$, or λ_r will from a certain step r_0, i. e. for $r \geqq r_0$, be constant $= \Omega$.

We define now for every number σ in the interval $\Omega < \sigma < \infty$ the number $r = \psi(\sigma)$ as the greatest lower bound of those values $r' \geqq 0$ for which $\lambda_{r'} \leqq \sigma$. The function $r = \psi(\sigma)$ is called the *summability function* of the Dirichlet series. It is equal to 0 for $\sigma \geqq \lambda_0$ and in the interval $\Omega < \sigma \leqq \lambda_0$ (when we suppose that $\Omega < \lambda_0$) it is simply

[1] This paper is based on notes left by Professor Harald Bohr. The manuscript has been prepared by Dr. Erling Følner.

the inverse function of $\sigma = \lambda_r$. Hence it follows from the above results that $r = \psi(\sigma)$ is a continuous convex function which in the interval $\Omega < \sigma \leqq \lambda_0$ is strictly decreasing with numerical slope $\geqq 1$, i.e. with a left derivative $\psi'(\lambda_0 - 0) \leqq -1$ at the point $\sigma = \lambda_0$. Further, if λ_r is constant $= \Omega$ from a certain step r_0, then $\psi(\sigma) \rightarrow r_0$ for $\sigma \rightarrow \Omega$; otherwise $\psi(\sigma) \rightarrow \infty$ for $\sigma \rightarrow \Omega$.

Contrary to the abscissa of convergence λ_0, the abscissa Ω has a simple function theoretical meaning (H. Bohr [2], and [3], p. 124; M. Riesz [7]). Indeed, for every $\sigma_0 > \Omega$, the function $f(s)$ represented by the series is of finite order with respect to t in the half plane $\sigma > \sigma_0$, i.e. there exists a number $l \geqq 0$ such that

$$(2) \qquad f(\sigma + it) = O(|t|^l)$$

when $|t| \rightarrow \infty$, uniformly for all $\sigma > \sigma_0$, whereas $f(s)$ is not regular and of finite order in any half plane $\sigma > \sigma_0$ where $\sigma_0 < \Omega$. For every $\sigma > \Omega$ we define the number $\mu(\sigma)$ as the greatest lower bound of those values $l \geqq 0$ for which (2) holds for this value of σ. This function $\mu(\sigma)$ is called the *order function*, or the Lindelöf μ-function, of $f(s)$. It is equal to 0 for $\sigma > \lambda_0 + 1$ because the Dirichlet series is absolutely convergent for $\sigma > \lambda_0 + 1$. It follows from the Phragmén-Lindelöf theorem that the function $\mu(\sigma)$ is a continuous convex function. Thus, denoting by ω_μ ($\leqq \lambda_0 + 1$) the smallest number with the property that $\mu(\sigma) = 0$ for $\sigma \geqq \omega_\mu$, the function $\mu(\sigma)$ is (when $\Omega < \omega_\mu$) strictly decreasing in the interval $\Omega < \sigma \leqq \omega_\mu$. We mention that $\mu(\sigma_0)$ $(\Omega < \sigma_0 < \infty)$ is also the order of $f(s)$ in the half plane $\sigma > \sigma_0$, i.e. the greatest lower bound of those values $l \geqq 0$ for which (2) holds uniformly for all $\sigma > \sigma_0$.

As to the connection between $\psi(\sigma)$ and $\mu(\sigma)$ it is known (see [6], pp. 49 and 53) that

$$\psi(\sigma) \leqq \mu(\sigma) \leqq \psi(\sigma) + 1.$$

The present paper deals with the problem whether the above results concerning the functions $\psi(\sigma)$ and $\mu(\sigma)$ and the connection between them are the best possible, i.e. whether conversely for two functions $\psi(\sigma)$ and $\mu(\sigma)$ which have all the properties mentioned above there exists an ordinary Dirichlet series $f(s) = \sum a_n n^{-s}$ with $\psi(\sigma)$ as summability function and $\mu(\sigma)$ as order function. No complete answer is obtained, but it is shown that if we impose on the function $\mu(\sigma)$ the additional condition that it, too, has a numerical slope $\geqq 1$ in the interval in which it is strictly decreasing, i.e. that (when $\Omega < \omega_\mu$) we have $\mu'(\omega_\mu - 0) \leqq -1$, then the answer is in the affirmative. In other words, we shall prove the following

Main Theorem. *Let $\psi(\sigma)$ be a continuous convex function defined in an interval $\sigma > \Omega\ (\geqq -\infty)$ and equal to 0 to the right of a certain finite abscissa $\omega_\psi \geqq \Omega$ and (if $\omega_\psi > \Omega$) such that $\psi'(\omega_\psi - 0) \leqq -1$. Further, let $\mu(\sigma)$ be a continuous convex function defined in the same interval $\sigma > \Omega$ and equal to 0 to the right of a certain finite abscissa $\omega_\mu \geqq \Omega$ and (if $\omega_\mu > \Omega$) such that $\mu'(\omega_\mu - 0) \leqq -1$. Finally, let*

$$\psi(\sigma) \leqq \mu(\sigma) \leqq \psi(\sigma) + 1$$

for all $\sigma > \Omega$.

Then there exists a Dirichlet series $f(s) = \sum a_n n^{-s}$ which has the given functions $\psi(\sigma)$ and $\mu(\sigma)$ as summability function and order function, respectively.

We remark that as a consequence of the assumptions of the theorem we have $\omega_\psi \leqq \omega_\mu \leqq \omega_\psi + 1$. The condition $\omega_\mu \leqq \omega_\psi + 1$, which according to the above results is necessary whether $\mu'(\omega_\mu - 0) \leqq -1$ or not, therefore has not been included in the theorem.

We do not know whether there exist ordinary Dirichlet series $f(s) = \sum a_n n^{-s}$ for which the order function $\mu(\sigma)$ is not identically zero and does not satisfy the condition $\mu'(\omega_\mu - 0) \leqq -1$. For the zeta-series with alternating signs

$$\zeta(s)(1 - 2^{1-s}) = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s}$$

it is known that $\mu(\sigma) = 0$ for $\sigma \geqq 1$ and $\mu(\sigma) = \frac{1}{2} - \sigma$ for $\sigma \leqq 0$. The question as to whether $\mu'(\omega_\mu - 0) \leqq -1$ therefore amounts to whether $\mu\left(\frac{1}{2}\right) = 0$ $\left(\text{and hence } \mu(\sigma) = 0 \text{ for } \sigma \geqq \frac{1}{2} \text{ and } \mu(\sigma) = \frac{1}{2} - \sigma \text{ for } \sigma \leqq \frac{1}{2}\right)$, i.e. to the Lindelöf hypothesis $\zeta\left(\frac{1}{2} + it\right) = O(|t|^\varepsilon)$ for every $\varepsilon > 0$.

If we restrict our attention to the summability function $\psi(\sigma)$, we immediately see from the Main Theorem, that the known results are the best possible, i.e. any continuous convex function $\psi(\sigma)$ defined in an interval $\sigma > \Omega (\geq -\infty)$ and equal to 0 to the right of a certain finite abscissa $\omega_\psi \geq \Omega$ and (if $\omega_\psi > \Omega$) such that $\psi'(\omega_\psi - 0) \leq -1$, is the summability function of an ordinary Dirichlet series. Indeed, we have only to apply the Main Theorem, choosing $\mu(\sigma) = \psi(\sigma)$. This result generalizes a result of the author ([3], pp. 104—110) concerning the abscissae of summability of integral order, according to which the inequalities (1) are the best possible.

In the proof of the Main Theorem certain basic examples play a decisive role. In these examples $\Omega = -\infty$ (so that we are dealing with entire functions) and the ψ-curve as well as the μ-curve are half lines as soon as they have left the real axis, i.e. in the intervals $-\infty < \sigma < \omega_\psi$ and $-\infty < \sigma < \omega_\mu$, respectively. It appears immediately from the above inequalities that these half lines must be parallel and that the μ-line must lie above or coincide with the ψ-line. Further, their distance measured on a vertical line must be ≤ 1. Our basic examples correspond to those extreme cases where the two half lines coincide or have the vertical distance 1. In the special case where the numerical slope a of the half lines has its minimum value $a = 1$ examples have already been constructed by the author ([4], pp. 10—14, and [5], pp. 713—720). Generalizing these examples we construct in § 2 and § 3 examples for an arbitrary $a \geq 1$. (The reader need not know the examples for $a = 1$.)

In § 4 we construct from the extreme cases in § 2 and § 3 all intermediate cases where still both the ψ-curve and the μ-curve are half lines to the left of ω_ψ and ω_μ, respectively. The Dirichlet series obtained in § 4 are to serve as our "bricks" in the final construction in § 6 in which a Dirichlet series is formed by linear combination of denumerably many such series. § 5 is inserted for the purpose of giving two lemmas concerning the summability function and the order function of a Dirichlet series obtained by linear combination of denumerably many Dirichlet series.

§ 2. Construction, for an arbitrary $a \geqq 1$, of a Dirichlet series with $\psi(\sigma) = \mu(\sigma) = \begin{cases} 0 & \text{for } \sigma \geqq 0 \\ -a\sigma & \text{for } \sigma \leqq 0. \end{cases}$

Let $p_1, p_2, p_3, \cdots$ be a sequence of positive integers which satisfy the condition

$$(1) \qquad p_{m+1} \geqq (m+1)\, p_m$$

for all m and let

$$d_m = [p_m^{1-\theta}],$$

where for brevity's sake we have put $\frac{1}{a} = \theta$ $(0 < \theta \leqq 1)$. We consider the Dirichlet series

$$\sum_{n=1}^{\infty} a_n n^{-s} = p_1^{-s} - (p_1 + d_1)^{-s} + p_2^{-s} - 2\,(p_2 + d_2)^{-s} + (p_2 + 2\,d_2)^{-s} + \cdots$$

$$+ p_m^{-s} - \binom{m}{1}(p_m + d_m)^{-s} + \binom{m}{2}(p_m + 2\,d_m)^{-s} - \cdots + (-1)^m \binom{m}{m}(p_m + m d_m)^{-s}$$

$$+ \cdots = \sum_{m=1}^{\infty} \Delta_{d_m}^m (p_m^{-s}).$$

Here we have used the notation $\Delta_d^m u_p$ for the m^{th} difference with span d, i. e.

$$\Delta_d^m u_p = u_p - \binom{m}{1} u_{p+d} + \binom{m}{2} u_{p+2d} - \cdots + (-1)^m \binom{m}{m} u_{p+md}.$$

For such differences we shall use the known inequality (see for instance H. Bohr [4], p. 15)

$$(2) \qquad |\Delta_d^m (p^{-s})| \leqq 2^{m-h} |s|\,|s+1| \cdots |s+h-1|\, d^h p^{-\sigma-h},$$

which is valid for $d > 0$, $p > 0$, $\sigma + h > 0$, and $h = 0, 1, 2, \cdots, m$.

The above series has previously been considered by the author ([3], pp. 94—99), and it was shown that its abscissae of summability λ_h of integral order h are determined by

$$\lambda_h = -h\theta \quad (h = 0, 1, 2, \cdots).$$

Thus $\Omega = -\infty$, and $\psi(-h\theta) = h$ for $h = 0, 1, 2, \cdots$. Since $\psi(\sigma)$ is convex, this implies that $\psi(\sigma) = -a\sigma$ for $\sigma \leqq 0$, and hence

$\psi(\sigma) = 0$ for $\sigma \geqq 0$. Thus it only remains to prove that $\mu(\sigma) = -a\sigma$ for $\sigma \leqq 0$, which implies that $\mu(\sigma) = 0$ for $\sigma \geqq 0$.

Since $\mu(\sigma) \geqq \psi(\sigma)$ it is enough to show that $\mu(\sigma) \leqq -a\sigma$ for $\sigma \leqq 0$. Further, in order to prove this latter relation it suffices to prove that

$$(3) \qquad f(s) = O(|t|^h) \quad \text{for} \quad \sigma = -h\theta + \varepsilon,$$

where h runs through the numbers $0, 1, 2, \cdots$ and $\varepsilon > 0$ is arbitrary. Indeed, the inequality $\mu(-h\theta + \varepsilon) \leqq h$ together with the continuity of $\mu(\sigma)$ implies that $\mu(-h\theta) \leqq h$, and this latter inequality for $h = 0, 1, 2, \cdots$ together with the convexity of $\mu(\sigma)$ implies that $\mu(\sigma) \leqq -a\sigma$ for all $\sigma \leqq 0$.

In the proof of (3) we shall use the fact that $\sum 2^m p_m^{-\varepsilon}$ is convergent for every $\varepsilon > 0$. This fact, however, follows at once from (1) in view of which

$$\frac{2^{m+1} p_{m+1}^{-\varepsilon}}{2^m p_m^{-\varepsilon}} \to 0 \quad \text{for} \quad m \to \infty.$$

We write

$$f(s) = \sum_{m=1}^{h} \Delta_{d_m}^m (p_m^{-s}) + \sum_{m=h+1}^{\infty} \Delta_{d_m}^m (p_m^{-s}),$$

where the sum $f_1(s) = \sum_{m=1}^{h} \Delta_{d_m}^m (p_m^{-s})$ consists only of a finite number of terms $a_n n^{-s}$ and therefore is bounded on every line $\sigma = \sigma_0$. In the series $\sum_{m=h+1}^{\infty} \Delta_{d_m}^m (p_m^{-s})$ we shall apply the above inequality (2) to each of the terms $\Delta_{d_m}^m (p_m^{-s})$, $m = h+1, h+2, \cdots$. We obtain for $m > h$ and s on the line $\sigma = -h\theta + \varepsilon$ (where $\sigma + h > 0$)

$$|\Delta_{d_m}^m (p_m^{-s})| \leqq 2^{m-h} |s| |s+1| \cdots |s+h-1| d_m^h p_m^{-\sigma-h} \leqq$$

$$2^{-h} |s| |s+1| \cdots |s+h-1| 2^m p_m^{h(1-\theta)+h\theta-\varepsilon-h} =$$

$$2^{-h} |s| |s+1| \cdots |s+h-1| 2^m p_m^{-\varepsilon}.$$

Since $\sum 2^m p_m^{-\varepsilon}$ is convergent we see that $\sum_{m=h+1}^{\infty} \Delta_{d_m}^m (p_m^{-s})$ converges absolutely for $\sigma = -h\theta + \varepsilon$ and that its sum $f_2(s)$ satisfies the relation $f_2(s) = O(|t|^h)$. Finally, since $f(s) = f_1(s) + f_2(s)$, we see that $f(s) = O(|t|^h)$ for $\sigma = -h\theta + \varepsilon$, as we had to prove.

§ 3. Construction, for an arbitrary $a \geqq 1$, of a Dirichlet series with

$$\psi(\sigma) = \begin{cases} 0 & \text{for } \sigma \geqq 0 \\ -a\sigma & \text{for } \sigma \leqq 0 \end{cases} \quad \text{and} \quad \mu(\sigma) = \begin{cases} 0 & \text{for } \sigma \geqq \frac{1}{a} \\ 1 - a\sigma & \text{for } \sigma \leqq \frac{1}{a}. \end{cases}$$

In view of the general properties of the summability function and the order function it suffices to show that the constructed series has the right order function and the abscissa of convergence $\lambda_0 = 0$. Thus our task is to construct a Dirichlet series with $\lambda_0 = 0$ and $\Omega = -\infty$ and with the given function $\mu(\sigma)$ as order function.

We start again with a sequence of positive integers $p_1 < p_2 < p_3 < \cdots$ which increase rapidly. We assume here that they increase so rapidly that $\sum 2^m p_m^{-\varepsilon}$ converges for every $\varepsilon > 0$ and so that

$$\sum_{m=M+1}^{\infty} 2^m p_m^{-\varepsilon} = o(p_M^{-\varepsilon}) \text{ and } 2^{M-1} p_{M-1}^{L} = o(p_M^{\varepsilon})$$

for $M \to \infty$ and every $\varepsilon > 0$ and $L > 0$. Next, we choose integers l_m and d_m of the orders of magnitude p_m^a and p_m^{a-1}, respectively. It will be convenient to choose

$$l_m = [p_m^a] + 1 \quad \text{and} \quad d_m = [p_m^{a-1}].$$

Further, we put

$$t_m = \pi p_m$$

and choose the numbers q_m of a slightly smaller order of magnitude than the p_m. We set

$$q_m = \left[\frac{p_m}{(m+1)^3}\right].$$

We remark that the p_m from the beginning must be chosen so that certain inequalities which on account of the above demands are fulfilled for large m will be fulfilled for all m. The inequalities to which we refer (we shall not write them out explicitly) are those which express that the term groups given by the braces $\{\cdots\}_m$ in the series immediately below do not overlap.

Our Dirichlet series $f(s) = \sum a_n n^{-s}$ is now constructed from term groups $\{\cdots\}_m$ $(m = 1, 2, \cdots)$, the m^{th} term group of which consists of altogether $(q_m+1)(m+1)$ terms $a_n n^{-s}$. These terms are distributed in q_m+1 smaller term groups $[\cdots]_{m,\nu}$ $(\nu = 0, 1, 2, \cdots, q_m)$ each of which apart from a complex sign is simply an m^{th} difference (with span d_m) and thus contains $m+1$ terms $a_n n^{-s}$. More specifically, our series is defined in the following way:

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} = \sum_{m=1}^{\infty} \{\cdots\}_m = \sum_{m=1}^{\infty} \sum_{\nu=0}^{q_m} [\cdots]_{m,\nu}$$

where for $m \geq 1$, $0 \leq \nu \leq q_m$ the square bracket $[\cdots]_{m,\nu}$ has the meaning

$$[\cdots]_{m,\nu} = \left(l_m + \nu(m+1)d_m\right)^{il_m} \Delta_{d_m}^m \left(l_m + \nu(m+1)d_m\right)^{-s} =$$

$$\left(l_m + \nu(m+1)d_m\right)^{il_m} \left(\left(l_m + \nu(m+1)d_m\right)^{-s} - \binom{m}{1}\left(l_m + \nu(m+1)d_m + d_m\right)^{-s} + \cdots + (-1)^m \binom{m}{m}\left(l_m + \nu(m+1)d_m + md_m\right)^{-s}\right).$$

We shall now prove that this series $\sum a_n n^{-s}$ possesses all the desired properties. We divide the proof into three steps.

1°. We prove first that our series has the abscissa of convergence $\lambda_0 = 0$. Since $|a_n| = |l_m^{il_m}| = 1$ for $n = l_m (m = 1, 2, \cdots)$ we see that the series is divergent at the point $s = 0$ and it is consequently plain that $\lambda_0 \geq 0$. In order to show that $\lambda_0 \leq 0$, i. e. that the series is convergent for $\sigma > 0$, we first show that our series is absolutely convergent for $\sigma > 0$ when we preserve the square brackets (but not the braces). On account of a later application we shall even show that under preservation of the square brackets the series is absolutely convergent in the whole plane. We do this by showing the absolute convergence in the half plane $\sigma > \sigma_h = \dfrac{-h+1}{\alpha}$ for $h = 0, 1, 2, \cdots$. We write

$$f(s) = \sum_{m=1}^{\infty} \{\cdots\}_m = \sum_{m=1}^{h} \{\cdots\}_m + \sum_{m=h+1}^{\infty} \{\cdots\}_m,$$

where the first sum $\sum\limits_{m=1}^{h} \{\cdots\}_m$ only contains a finite number of square brackets $[\cdots]_{m,\nu}$. In order to prove that the second sum $\sum\limits_{m=h+1}^{\infty} \{\cdots\}_m$ is absolutely convergent when we keep the square brackets (but not the braces) we estimate each of the q_m+1 brackets $[\cdots]_{m,\nu}$ in the term $\{\cdots\}_m$ with index $m>h$ by the inequality (2), § 2. For $m>h$ and s in the half plane $\sigma>\sigma_h$ (where a fortiori $\sigma+h>0$) we get

$$|[\cdots]_{m,\nu}| = |\varDelta^m_{d_m}(l_m+\nu(m+1)d_m)^{-s}|$$
$$\leqq 2^{m-h}|s||s+1|\cdots|s+h-1|d^h_m l_m^{-\sigma-h}.$$

Hence the sum of the absolute values of the q_m+1 brackets $[\cdots]_{m,\nu}$ in $\{\cdots\}_m$ is estimated by

$$\sum_{\nu=0}^{q_m}|[\cdots]_{m,\nu}|\leqq(q_m+1)\,2^{m-h}|s||s+1|\cdots|s+h-1|d^h_m l_m^{-\sigma-h}$$

and consequently, since $q_m<p_m$, $d_m\leqq p_m^{a-1}$, and $l_m>p_m^a$ by

$$(1)\quad \sum_{\nu=0}^{q_m}|[\cdots]_{m,\nu}|\leqq 2^{-h}|s||s+1|\cdots|s+h-1|\,2^m p_m^{-a\sigma-h+1},$$

where $\sigma>\sigma_h$ and $m>h$. From this inequality we immediately infer the stated absolute convergence in the half plane $\sigma>\sigma_h=\dfrac{-h+1}{a}$; in fact, the series $\sum 2^m p_m^{-a\sigma-h+1}$ is convergent since the exponent $-a\sigma-h+1$ is smaller than $-a\sigma_h-h+1=0$. Thus, in order to show that the series $\sum a_n n^{-s}$ itself (i. e. the series without any brackets whatsoever) is convergent for $\sigma>0$, we only have to show that the partial sums of $[\cdots]_{m,\nu}$ for $\sigma>0$ tend to 0 for $m\to\infty$. That this is the case is, however, obvious since the sum of the absolute values of all the terms $a_n n^{-s}$ in $[\cdots]_{m,\nu}$ for $\sigma>0$ is

$$\leqq\sum_{j=0}^{m}\binom{m}{j}l_m^{-\sigma}=2^m l_m^{-\sigma}<2^m p_m^{-a\sigma},$$

which tends to 0 for $m\to\infty$.

2°. Next, we shall show that $\Omega=-\infty$ and $\mu(\sigma)\leqq 1-a\sigma$ for $\sigma\leqq\dfrac{1}{a}$. We first remark that it will suffice to show that $f(s)$ is regular for $\sigma>\sigma_h=\dfrac{-h+1}{a}$, where h runs through the numbers $0,1,2,\cdots$, and that

$$f(s)=O(|t|^h)\quad\text{for}\quad \sigma>\sigma_h+\varepsilon.$$

In fact, this will immediately imply that $\Omega = -\infty$ and

$$\mu(\sigma_h) \leq h = 1 - a\sigma_h \quad \text{for} \quad h = 0, 1, 2, \cdots,$$

and next, by help of the convexity of $\mu(\sigma)$ we get

$$\mu(\sigma) \leq 1 - a\sigma \quad \text{for all} \quad \sigma \leq \frac{1}{a}.$$

For $\sigma > \sigma_h$ we write again

$$f(s) = \sum_{m=1}^{h} \{\cdots\}_m + \sum_{m=h+1}^{\infty} \{\cdots\}_m.$$

The first sum $\sum_{m=1}^{h} \{\cdots\}_m$ contains only a finite number of terms $a_n n^{-s}$ and is therefore an entire function $f_1(s)$ bounded in every half plane $\sigma > \sigma_0$. In the second sum $\sum_{m=h+1}^{\infty} \{\cdots\}_m$ we estimate each of the terms $\{\cdots\}_m (m = h+1, h+2, \cdots)$ by the above inequality (1). For $m > h$ and s in the half plane $\sigma > \sigma_h + \varepsilon$ we get

$$|\{\cdots\}_m| \leq \sum_{\nu=0}^{q_m} |[\cdots]_{m,\nu}| \leq 2^{-h} |s| |s+1| \cdots |s+h-1| 2^m p_m^{-a\sigma - h + 1} \leq$$
$$2^{-h} |s| |s+1| \cdots |s+h-1| 2^m p_m^{-a\varepsilon}.$$

Since $\sum 2^m p_m^{-a\varepsilon}$ is convergent we infer that the infinite series $\sum_{m=h+1}^{\infty} \{\cdots\}_m$ is uniformly convergent in every bounded part of the half plane $\sigma > \sigma_h + \varepsilon$. Consequently, since $\varepsilon > 0$ is arbitrary, the function $f_2(s)$ represented by this series is regular in the half plane $\sigma > \sigma_h$; furthermore, it satisfies for $\sigma_h + \varepsilon < \sigma <$ (say) 2 (and hence of course also in the whole half plane $\sigma > \sigma_h + \varepsilon$) the inequality

$$f_2(s) = O(|t|^h).$$

Since $f(s)$ is obtained as the sum of $f_1(s)$ and $f_2(s)$, we see that $f(s)$ is regular for $\sigma > \sigma_h$ and equal to $O(|t|^h)$ for $\sigma > \sigma_h + \varepsilon$ as we had to prove.

3°. We come now to the salient point, namely the proof that $\mu(\sigma) \geq 1 - a\sigma$ for $\sigma < \frac{1}{a}$. Let σ_0 be an arbitrary abscissa $< \frac{1}{a}$;

we consider the behaviour of $f(s)$ at the points $s_M = \sigma_0 + it_M$ on the line $\sigma = \sigma_0$, where t_M are the previously introduced ordinates $t_M = \pi p_M$, and we shall (even) prove that for sufficiently large M

$$(2) \qquad |f(s_M)| > t_M^{1-a\sigma_0}.$$

For this purpose we first determine a positive integer h so that $\sigma_h = \frac{-h+1}{a} < \sigma_0 < \frac{1}{a}$. For $M > h$ we write

$$f(s_M) = \sum_{m=1}^{M-1} \{\cdots\}_m + \{\cdots\}_M + \sum_{m=M+1}^{\infty} \{\cdots\}_m = B_M(s_M) + T_M(s_M) + R_M(s_M),$$

and we shall prove that both the "beginning" $B_M(s_M)$ and the "remainder" $R_M(s_M)$ for $M \to \infty$ are equal to $o(t_M^{1-a\sigma_0})$ while the M^{th} term $T_M(s_M)$ for sufficiently large M is numerically larger than $2t_M^{1-a\sigma_0}$. In this way the inequality (2) will be proved.

(1) For the "beginning" $B_M(s_M)$ we use a rough estimate. The numerical value of each of its coefficients $a_n \neq 0$ is a binomial coefficient $\binom{m}{l}$ with $m \leq M-1$ and hence it is $\leq 2^{M-1}$. Thus

$$|B_M(s_M)| \leq 2^{M-1} \sum_{n=1}^{l'_{M-1}} n^{-\sigma_0},$$

where

$$l'_{M-1} = l_{M-1} + q_{M-1} M d_{M-1} + (M-1)\, d_{M-1} \leq 2\, l_{M-1}$$

for M sufficiently large. Hence, since $\sigma_0 < 1$,

$$|B_M(s_M)| \leq 2^{M-1} \sum_{n=1}^{2\,l_{M-1}} n^{-\sigma_0} = O(2^{M-1}\, l_{M-1}^{1-\sigma_0}) = O(2^{M-1}\, p_{M-1}^{a(1-\sigma_0)})$$

and consequently, since $1 - a\sigma_0 > 0$,

$$B_M(s_M) = o(p_M^{1-a\sigma_0}) = o(t_M^{1-a\sigma_0}).$$

(2) For the "remainder" $R_M(s_M)$ we can apply the inequality (1) since all occurring m are $> M > h$ and $\sigma_0 > \sigma_h$. We get, since $-a\sigma_0 - h + 1 < 0$,

$$|R_M(s_M)| \leq 2^{-h} |s_M|\, |s_M + 1| \cdots |s_M + h - 1| \sum_{m=M+1}^{\infty} 2^m p_m^{-a\sigma_0 - h + 1} =$$

$$O(t_M^h)\, o(p_M^{-a\sigma_0 - h + 1}) = O(t_M^h)\, o(t_M^{-a\sigma_0 - h + 1}) = o(t_M^{1-a\sigma_0}).$$

(3) We shall finally prove that the M^{th} term $T_M(s_M)$ satisfies the inequality

$$|T_M(s_M)| > 2\, t_M^{1-a\sigma_0}$$

for all sufficiently large M. The reason for the validity of this inequality is that all the terms $a_n n^{-s}$ occurring in $T_M(s_M)$ (and there are rather many of them on account of the choice of the q_m), namely the $(q_M+1)\ (M+1)$ terms distributed in the q_M+1 brackets $[\cdots]_{M,\nu}$ with $M+1$ terms in each bracket, for sufficiently large M "almost point in the same direction"; more precisely: these terms all lie in the angle $-\frac{\pi}{3} < v < \frac{\pi}{3}$. We postpone the verification of this fact for a moment and shall first show that when once this property is established we can immediately complete the proof. In fact, we may argue as follows. The sum of the binomial coefficients occurring in each of the q_M+1 square brackets is equal to 2^M, and every n occurring in the sum belongs to the interval $l_M \leq n \leq l'_M$ and a fortiori to the interval $l_M \leq n \leq 2\, l_M$ when M is large. Thus, for sufficiently large M we have

$$|T_M(s_M)| \geq \Re\, T_M(s_M) \geq \cos\frac{\pi}{3}(q_M+1)\, 2^M \operatorname*{Min}_{l_M \leq n \leq 2l_M} n^{-\sigma_0} \geq$$

$$\frac{1}{2}(q_M+1)\, 2^M \frac{1}{2} l_M^{-\sigma_0} > \frac{1}{2} 2^M \frac{p_M}{(M+1)^3} \frac{1}{2}\frac{1}{2} p_M^{-a\sigma_0} =$$

$$\frac{1}{8}\frac{2^M}{(M+1)^3} p_M^{1-a\sigma_0} = \frac{1}{8}\frac{2^M}{(M+1)^3}\left(\frac{t_M}{\pi}\right)^{1-a\sigma_0},$$

and this last quantity is larger than $2\, t_M^{1-a\sigma_0}$ for large M.

It remains to prove the decisive fact that all terms $a_n n^{-s}$ in $T_M(s_M)$ lie in the angle $-\frac{\pi}{3} < v < \frac{\pi}{3}$ for M sufficiently large. That this is the case is of course due to our choice of the complex signs of the occurring coefficients $a_n \neq 0$. We consider an arbitrary one of the q_M+1 brackets $[\cdots]_{M,\nu}$

$$\left(l_M + \nu(M+1)\, d_M\right)^{it_M} \Delta_{d_M}^M \left(l_M + \nu(M+1)\, d_M\right)^{-s_M} (\nu = 0, 1, 2, \cdots, q_M).$$

Denoting the number $l_M + \nu(M+1)\, d_M$ by $r = r(M, \nu)$ we get

$$[\cdots]_{M,\nu} = r^{it_M}\left(r^{-s_M} - \binom{M}{1}(r+d_M)^{-s_M} + \cdots + (-1)^M \binom{M}{M}(r+Md_M)^{-s_M}\right).$$

The amplitudes of the single terms

$$r^{it_M}(-1)^\lambda \binom{M}{\lambda}(r+\lambda d_M)^{-(\sigma_0+it_M)} \quad (\lambda = 0, 1, 2, \cdots, M)$$

are given by

$$-t_M \log(r+\lambda d_M) + t_M \log r + \lambda\pi = -t_M \log\left(1+\frac{\lambda d_M}{r}\right) + \lambda\pi =$$

$$-\lambda \frac{t_M d_M}{r} \frac{\log\left(1+\frac{\lambda d_M}{r}\right)}{\frac{\lambda d_M}{r}} + \lambda\pi.$$

When we take account of the fact that $0 \leq \lambda \leq M$ and $0 \leq \nu \leq q_M$ and insert the known expressions for t_M, d_M, r, q_M, we see at once that

$$\frac{t_M d_M}{r} \to \pi \quad \text{and} \quad \frac{\lambda d_M}{r} \to 0 \quad \text{for} \quad M \to \infty$$

independently of ν and λ. In view of this, the above formula for the amplitudes together with the relation

$$\lim_{x \to 0} x^{-1} \log(1+x) = 1$$

yields the result that the amplitudes of the single terms in $T_M(s_M)$ tend to 0 for $M \to \infty$. In particular, these amplitudes lie in the angle $-\frac{\pi}{3} < v < \frac{\pi}{3}$ for M sufficiently large.

Thus, all our statements concerning $f(s) = \sum a_n n^{-s}$ are proved.

§ 4. Our "bricks".

We shall now, for an arbitrary $a \geq 1$, construct a class of Dirichlet series for which again the ψ-curve and μ-curve when they have left the real axis are half lines with numerical slope a, but where the vertical distance from the ψ-half-line to the μ-half-line no longer assumes just one of its extreme values 0 or 1, but has an arbitrary value between these two limits. At the same time we shall perform a trivial translation in the direction of the real axis. For the sake of convenience, we characterize a function of σ which is 0 for $\omega \leq \sigma < \infty$ and equal to $-a(\sigma - \omega)$

for $-\infty < \sigma \leq \omega$ by the symbol $\{\omega; a\}$. We shall prove the following

Theorem. *For arbitrary ω, a, and d such that $a \geq 1$ and $0 \leq d \leq \frac{1}{a}$ there exists a Dirichlet series $f(s) = \sum a_n n^{-s}$ with the summability function $\{\omega; a\}$ and the order function $\{\omega + d; a\}$.*

In the proof we may evidently assume that $\omega = 0$. Also, we may assume that $0 < d < \frac{1}{a}$. We know that there exist two Dirichlet series $f_1(s) = \sum a'_n n^{-s}$ and $f_2(s) = \sum a''_n n^{-s}$, where the ψ- and μ-functions of the first series are given by

$$\psi_1 = \mu_1 = \{0; a\}$$

while the ψ- and μ-functions of the second series are given by

$$\psi_2 = \{0; a\} \quad \text{and} \quad \mu_2 = \left\{\frac{1}{a}; a\right\}.$$

We now replace s by $s + \frac{1}{a} - d$ in $f_2(s)$, i.e. we consider instead of $f_2(s)$ the function $f_3(s) = f_2\left(s + \frac{1}{a} - d\right) = \sum a'''_n n^{-s}$. The ψ- and μ-functions of $f_3(s)$ are given by

$$\psi_3 = \left\{d - \frac{1}{a}; a\right\} \quad \text{and} \quad \mu_3 = \{d; a\}.$$

We shall now show that the series

$$f(s) = f_1(s) + f_3(s) = \sum (a'_n + a'''_n) n^{-s} = \sum a_n n^{-s}$$

will satisfy our demands.

First, $\Omega = -\infty$. Secondly, the summability function $\psi(\sigma)$ is equal to 0 for $\sigma \geq 0$ since both $\sum a'_n n^{-s}$ and $\sum a'''_n n^{-s}$ are convergent for $\sigma > 0$, and $\psi(\sigma) = \psi_1(\sigma)$ for every negative σ since $\psi_1(\sigma) > \psi_3(\sigma)$. (We have used here the fact that the sum of two series of constant terms both of which are summable of the r^{th} order is again a series summable of the r^{th} order, while the sum of two series of which the one series is summable of the r^{th} order and the other is not, is a series which is not summable of the r^{th} order.) Thirdly, the order function $\mu(\sigma)$ is equal to 0 for $\sigma \geq d$ since both $\mu_1(\sigma)$ and $\mu_3(\sigma)$ are equal to 0 here, and

$\mu(\sigma) = \mu_3(\sigma)$ for $\sigma < d$ since $\mu_3(\sigma) > \mu_1(\sigma)$. (We have used here the fact that the sum of two functions of t which are both $O(|t|^h)$ is again $O(|t|^h)$ while the sum is not $O(|t|^h)$ if one of the functions is $O(|t|^h)$ and the other is not.)

§ 5. Two lemmas.

In this section we shall prove two lemmas concerning summability and order of magnitude of Dirichlet series which are formed by linear combination of infinitely many Dirichlet series.

Before passing to these theorems we start with the following

Remark. *Let*

$$f_0(s) = \sum a_n^{(0)} n^{-s}, \quad f_1(s) = \sum a_n^{(1)} n^{-s}, \cdots$$

be a sequence of Dirichlet series which we assume to be all absolutely convergent (at least) for $\sigma \geqq \sigma_0$. We assert that it is possible to determine a sequence of positive numbers $E_0, E_1, \cdots$ so that the infinite series

$$(1) \qquad \sum_{N=0}^{\infty} \varepsilon_N a_1^{(N)}, \quad \sum_{N=0}^{\infty} \varepsilon_N a_2^{(N)}, \cdots$$

are convergent for every sequence $\varepsilon_0, \varepsilon_1, \cdots$ with

$$0 < \varepsilon_0 < E_0, \quad 0 < \varepsilon_1 < E_1, \cdots$$

and that further, when the sums of these infinite series are denoted by $A_1, A_2, \cdots$, the series

$$(2) \qquad \varepsilon_0 f_0(s) + \varepsilon_1 f_1(s) + \cdots$$

and the Dirichlet series (obtained by formal calculation from (2))

$$(3) \qquad \sum_{n=1}^{\infty} A_n n^{-s}$$

will be absolutely convergent for $\sigma \geqq \sigma_0$ and have the same sum.

Proof. We put

$$\sum_{n=1}^{\infty} |a_n^{(0)}| n^{-\sigma_0} = K_0, \quad \sum_{n=1}^{\infty} |a_n^{(1)}| n^{-\sigma_0} = K_1, \cdots$$

and choose the positive numbers $E_0, E_1, \cdots$ so that the series $\sum E_n K_n$ is convergent. Then

$$\sum_{N,n} \varepsilon_N \,|\, a_n^{(N)} \,|\, n^{-\sigma} < \infty \quad \text{for} \quad \sigma \geqq \sigma_0$$

so that

$$\sum_{N=0}^{\infty} \varepsilon_N f_N(s) = \sum_{N=0}^{\infty} \varepsilon_N \sum_{n=1}^{\infty} a_n^{(N)} n^{-s} = \sum_{n=1}^{\infty} n^{-s} \sum_{N=0}^{\infty} \varepsilon_N a_n^{(N)} = \sum_{n=1}^{\infty} A_n n^{-s},$$

where all occurring series are absolutely convergent ($\sigma \geqq \sigma_0$). It is plain that the conclusion still holds (with the same E's) when we omit the assumption $0 < \varepsilon_N < E_N$ for finitely many indices N.

Lemma 1. *Let*

$$g_0(s) = \sum b_n^{(0)} n^{-s}, \; g_1(s) = \sum b_n^{(1)} n^{-s}, \; g_2(s) = \sum b_n^{(2)} n^{-s}, \cdots$$

be a sequence of Dirichlet series (each of which possesses a half plane of convergence). Denoting by $\Lambda_r^{(N)}$ *the* r^{th} *abscissa of summability of the function* $g_N(s)$ $(N = 0, 1, 2, \cdots)$ *we assume that there exists a number* $r\,(\geqq 0)$ *such that*

$$\Lambda_r^{(N)} < \Lambda_r^{(0)} = \Lambda \quad (N = 1, 2, \cdots).$$

It follows immediately that the ψ*-curve of all the Dirichlet series for* $\sigma > \Lambda$ *must lie under or on the curve* $\{\Lambda + r; 1\}$ *so that all the Dirichlet series must be absolutely convergent for* $\sigma > \Lambda + r + 1$, *in particular for* $\sigma \geqq \Lambda + r + 2$.

Then there exists a sequence of positive numbers $e_1 < E_1, e_2 < E_2, \cdots$ [*where* E_N $(N = 0, 1, 2, \cdots)$ *are obtained from the above remark applied to the functions* $g_N(s)$ $(N = 0, 1, 2, \cdots)$ *and* $\Lambda + r + 2$ *instead of* σ_0] *such that the Dirichlet series*

$$(4) \qquad G(s) = g_0(s) + \varepsilon_1 g_1(s) + \varepsilon_2 g_2(s) + \cdots = \sum B_n n^{-s},$$

where

$$(5) \qquad B_n = b_n^{(0)} + \varepsilon_1 b_n^{(1)} + \varepsilon_2 b_n^{(2)} + \cdots$$

for every sequence $\varepsilon_1, \varepsilon_2, \cdots$ *such that*

(6) $$0 < \varepsilon_1 < e_1,\ 0 < \varepsilon_2 < e_2, \cdots$$

will have its r^{th} abscissa of summability equal to the number Λ. (The series (5) converge and the two series in (4) are absolutely convergent for $\sigma \geqq \Lambda + r + 2$ with the same sum $G(s)$. This follows immediately from the above remark since $e_N < E_N$ for $N = 1, 2, \cdots$.)

Proof. We have to prove that we can choose the positive numbers $e_N < E_N$ so that the series $G(s) = \sum B_n n^{-s}$ under the assumption (6) is summable of the r^{th} order for $\sigma > \Lambda$, but not summable of the r^{th} order for any $\sigma < \Lambda$. We divide the proof into two parts.

1°. In this part we choose the positive numbers $e_N < E_N$ so that the series $G(s) = \sum B_n n^{-s}$ under the assumption (6) is summable of the r^{th} order for $\sigma > \Lambda$. In order to obtain this result, it is obviously enough to secure that the series

$$G^*(s) = \varepsilon_1 g_1(s) + \varepsilon_2 g_2(s) + \cdots = \sum B_n^* n^{-s}$$

becomes summable of the r^{th} order at the point $s = \Lambda$; for when both of the series $g_0(s) = \sum b_n^{(0)} n^{-s}$ and $G^*(s) = \sum B_n^* n^{-s}$ are summable of the r^{th} order for $\sigma > \Lambda$, then their sum $G(s) = \sum B_n n^{-s}$ will have the same property. In the proof we shall suppose that $\Lambda = 0$. This is of course no real limitation since when $\Lambda \neq 0$ we may replace s by $s + \Lambda$. Since the abscissae of summability $\Lambda_r^{(1)}, \Lambda_r^{(2)}, \cdots$ are all smaller than Λ, the series $\sum b_n^{(1)}, \sum b_n^{(2)}, \cdots$ are all summable of the r^{th} order. We have

(7) $$B_n^* = \sum_{N=1}^{\infty} \varepsilon_N b_n^{(N)} \text{ (convergent for } \varepsilon_N < E_N).$$

In our proof we make use of the fact (see [6], pp. 21—22) that a series $\sum_{n=1}^{\infty} a_n$ is summable of a given order r if and only if a certain linear expression $S_n = \sum_{\nu=1}^{n} k_\nu a_\nu$ in the first n terms of the series (with coefficients k_ν which depend not only on ν but also on n and r) tends to a limit, the summability value of the series, for $n \to \infty$. We denote the expression S_n for the series $\sum b_n^{(1)}$,

$\sum b_n^{(2)}, \cdots$ by $T_n^{(1)}, T_n^{(2)}, \cdots$, respectively, and the expression S_n for the series $\sum B_n^*$ by T_n^*. Then from (7) it follows that

$$(8) \qquad T_n^* = \varepsilon_1 T_n^{(1)} + \varepsilon_2 T_n^{(2)} + \cdots.$$

Here, the quantity $T_n^{(N)}$ will for each $N = 1, 2, \cdots$ tend to a limit $U^{(N)}$, the summability value of the series $\sum_n b_n^{(N)}$, when $n \to \infty$. Hence there exist constants K_N such that

$$|T_n^{(N)}| \leq K_N \quad (n = 1, 2, \cdots).$$

We now choose the positive numbers $e_N < E_N$ so that

$$\sum_{N=1}^{\infty} e_N K_N$$

converges; then for every choice of the numbers ε_N in the intervals $0 < \varepsilon_N < e_N$ the series (8) will be uniformly convergent in n since it is majorized by $\sum e_N K_N$. Since each of the terms $\varepsilon_N T_n^{(N)}$ tends to a limit for $n \to \infty$ (namely $\varepsilon_N U^{(N)}$) it follows that the sum T_n^* of the series will also tend to a limit for $n \to \infty$ (namely $U^* = \varepsilon_1 U^{(1)} + \varepsilon_2 U^{(2)} + \cdots$), as we had to prove.

2°. In this part we choose the positive numbers $e_N < E_N$ so that the series $G(s) = \sum B_n n^{-s}$ under the assumption (6) is not summable of the r^{th} order for any $\sigma < A$, i.e. so that the r^{th} abscissa of summability is $\geq A$. If the series $g_0(s) = \sum b_n^{(0)} n^{-s}$ (with the r^{th} abscissa of summability A) is not summable of the r^{th} order at the point $s = A$ we can use the numbers e_N found under 1°. In fact, we saw that $G^*(s) = \sum B_n^* n^{-s}$ under the assumption (6) is summable of the r^{th} order at the point $s = A$ so that the series $\sum B_n n^{-s}$, which arises by termwise addition of $\sum b_n^{(0)} n^{-s}$ and $\sum B_n^* n^{-s}$, cannot be summable of the r^{th} order at the point $s = A$ and therefore must have its r^{th} abscissa of summability $\geq A$. However, we have not made this special assumption concerning the series $\sum b_n^{(0)} n^{-s}$ and as a matter of fact we could not make it in view of the applications. Hence we must proceed differently, and we shall use the known expression for the r^{th} abscissa of summability λ_r of a Dirichlet series $\sum a_n n^{-s}$ by means of the coefficients of the series. In the proof we shall assume that the number $\varLambda$ is > 0, say $= 1$ (since the expression just mentioned is only valid when $\lambda_r > 0$). This is of course no real limitation since when $\varLambda \neq 1$ we may replace s by $s - \lambda$, where $\lambda + \varLambda = 1$. As to this expression of λ_r by the coefficients of the series we shall only use the following fact (see [6], p. 45 and [3], p. 86 and [1], pp. 70—71). There exists a linear expression $S_n = \sum_{\nu=1}^{n} k_\nu a_\nu$ in the first n coefficients of the series (with coefficients k_ν, which depend not only on ν but also on n and r) such that the necessary and sufficient condition in order that the series $\sum a_n n^{-s}$ have its r^{th} abscissa of summability $\lambda_r \geqq 1$ is that

S_n is not equal to $O(n^{-\delta})$ for any $\delta > 0$, or equivalently
S_n is not equal to $o(n^{-\delta})$ for any $\delta > 0$.

(The expression S_n here is not, of course, the same as the expression S_n under 1°.)

We shall denote the expressions S_n corresponding to the series $\sum b_n^{(0)} n^{-s}, \sum b_n^{(1)} n^{-s}, \cdots$ by $T_n^{(0)}, T_n^{(1)}, \cdots$, respectively, and the expression S_n for the series $\sum B_n n^{-s}$ by T_n. Since by assumption the series $\sum b_n^{(0)} n^{-s}$ has its r^{th} abscissa of summability $= \varLambda = 1$ we know that to any given $\delta > 0$ there exist infinitely many values of n for which

$$| T_n^{(0)} | > n^{-\delta}.$$

Since each of the series $\sum_n b_n^{(N)} n^{-s}$ $(N = 1, 2, \cdots)$ has its r^{th} abscissa of summability $\varLambda_r^{(N)} < \varLambda = 1$ there exists for every $N = 1, 2, \cdots$ a number $\varDelta_N > 0$ such that

$$T_n^{(N)} = o(n^{-\varDelta_N}).$$

It suffices to show that T_n for a suitable choice of the positive constants $e_N < E_N$ under the assumption (6) for every $\delta > 0$ satisfies the inequality

$$| T_n | > \frac{1}{3} n^{-\delta}$$

for infinitely many values of n. This is equivalent to saying that for some sequence $\delta_1, \delta_2, \cdots$ of positive numbers which tends to 0 there must exist a corresponding sequence of positive integers $n_1 < n_2 < \cdots$ such that the inequality

$$| T_{n_m} | > \frac{1}{3} n_m^{-\delta_m}$$

is satisfied for all $m = 1, 2, \cdots$. As δ-sequence we shall here use an arbitrary sequence of positive numbers which tends to 0 and satisfies the conditions

$$\delta_2 \leq \Delta_1, \quad \delta_3 \leq \min(\Delta_1, \Delta_2), \cdots.$$

We shall now indicate positive numbers $e_N < E_N$ with the desired properties. We proceed in steps.

First step. We choose a positive integer n_1 so that

$$| T_{n_1}^{(0)} | > n_1^{-\delta_1}.$$

For this $n = n_1$ the expressions $T_n^{(1)}, T_n^{(2)}, \cdots$ assume certain values, say $k_{11}, k_{12}, \cdots$. We choose the positive numbers $e_{11} < E_1$, $e_{12} < E_2, \cdots$ so that

$$\sum_{N=1}^{\infty} e_{1N} | k_{1N} |$$

is convergent with sum $< \frac{1}{2} n_1^{-\delta_1}$. On the analogy of (8) we have

$$T_n = T_n^{(0)} + \varepsilon_1 T_n^{(1)} + \varepsilon_2 T_n^{(2)} + \cdots \quad (\text{for } 0 < \varepsilon_N < E_N). \tag{9}$$

Hence, for every choice of $\varepsilon_1, \varepsilon_2, \cdots$ in the intervals $0 < \varepsilon_1 < e_{11}$, $0 < \varepsilon_2 < e_{12}, \cdots$ we have

$$| T_{n_1} | \geq | T_{n_1}^{(0)} | - \sum_{N=1}^{\infty} \varepsilon_N | T_{n_1}^{(N)} | > n_1^{-\delta_1} - \sum_{N=1}^{\infty} e_{1N} | k_{1N} | > \frac{1}{2} n_1^{-\delta_1}.$$

Second step. We choose an integer $n_2 > n_1$ so that

$$| T_{n_2}^{(0)} | > n_2^{-\delta_2} \quad \text{and also} \quad E_1 | T_{n_2}^{(1)} | < \frac{1}{3} n_2^{-\delta_2}.$$

The latter inequality may be obtained since $\delta_2 \leq \Delta_1$. For this $n = n_2$ the expressions $T_n^{(2)}, T_n^{(3)}, \cdots$ assume certain values, say $k_{22}, k_{23}, \cdots$. We choose the positive numbers $e_{22} < E_2, e_{23} < E_3, \cdots$ so that

$$\sum_{N=2}^{\infty} e_{2N} | k_{2N} |$$

is convergent with sum $<\frac{1}{3}n_2^{-\delta_2}$. Then for every choice of $\varepsilon_1, \varepsilon_2, \cdots$ in the intervals $0<\varepsilon_1<E_1$, $0<\varepsilon_N<e_{2N}$ $(N=2,3,\cdots)$ we have, on account of (9),

$$|T_{n_2}| \geqq |T_{n_2}^{(0)}| - E_1 |T_{n_2}^{(1)}| - \sum_{N=2}^{\infty} e_{2N} |k_{2N}| >$$
$$n_2^{-\delta_2} - \frac{1}{3}n_2^{-\delta_2} - \frac{1}{3}n_2^{-\delta_2} = \frac{1}{3}n_2^{-\delta_2}.$$

. . .

m^{th} *step.* We choose an integer $n_m > n_{m-1}$ so that

$$|T_{n_m}^{(0)}| > n_m^{-\delta_m}$$

and also

$$E_1 |T_{n_m}^{(1)}| + E_2 |T_{n_m}^{(2)}| + \cdots + E_{m-1} |T_{n_m}^{(m-1)}| < \frac{1}{3} n_m^{-\delta_m}.$$

The latter inequality may be obtained since $\delta_m \leqq \min(\varDelta_1, \cdots, \varDelta_{m-1})$. For this $n = n_m$ the expressions $T_n^{(m)}, T_n^{(m+1)}, \cdots$ assume certain values, say $k_{mm}, k_{m,m+1}, \cdots$. We choose the positive numbers $e_{mm} < E_m$, $e_{m,m+1} < E_{m+1}, \cdots$ so that

$$\sum_{N=m}^{\infty} e_{mN} |k_{mN}|$$

is convergent with sum $<\frac{1}{3}n_m^{-\delta_m}$. Then for every choice of $\varepsilon_1, \varepsilon_2, \cdots$ in the intervals $0<\varepsilon_1<E_1, \cdots, 0<\varepsilon_{m-1}<E_{m-1}$, $0<\varepsilon_N<e_{mN}$ $(N=m, m+1, \cdots)$ we have, on account of (9),

$$|T_{n_m}| \geq |T_{n_m}^{(0)}| - \left(E_1 |T_{n_m}^{(1)}| + \cdots + E_{m-1} |T_{n_m}^{(m-1)}|\right) - \sum_{N=m}^{\infty} e_{mN} |k_{mN}| >$$
$$n_m^{-\delta_m} - \frac{1}{3}n_m^{-\delta_m} - \frac{1}{3}n_m^{-\delta_m} = \frac{1}{3}n_m^{-\delta_m}.$$

. . .

It appears from the above that the numbers

$$e_N = \min\{e_{1N}, \cdots, e_{NN}\} \quad (N = 1, 2, \cdots)$$

may be used to satisfy our demands under 2°.

Finally, for each N we choose the smaller one of the two numbers e_N found under 1° and 2° as our final e_N. These e_N satisfy the demands in Lemma 1. This completes the proof of Lemma 1.

Lemma 2. *Let*

$$h_0(s) = \sum c_n^{(0)} n^{-s}, \quad h_1(s) = \sum c_n^{(1)} n^{-s}, \quad h_2(s) = \sum c_n^{(2)} n^{-s}, \cdots$$

be a sequence of Dirichlet series (each of which possesses a half plane of convergence). We assume that all the functions $h_N(s)$ are regular and of finite order in a certain half plane $\sigma > \sigma_0$; further, denoting their orders of magnitude in this half plane by μ_N, we assume that

$$\mu_N < \mu_0 = \mu \quad \text{for} \quad N = 1, 2, \cdots.$$

It follows immediately that the ψ-curves of the Dirichlet series for $\sigma > \sigma_0$ must lie under or on the curve $\{\sigma_0 + \mu; 1\}$ so that all the Dirichlet series must be absolutely convergent for $\sigma > \sigma_0 + \mu + 1$, in particular for $\sigma \geqq \sigma_0 + \mu + 2$.

Then there exists a sequence of positive numbers $e_1 < E_1, e_2 < E_2, \cdots$ [where $E_N (N = 0, 1, 2, \cdots)$ are obtained from the previous remark applied to the functions $h_N(s)$ $(N = 0, 1, 2, \cdots)$ and $\sigma_0 + \mu + 2$ instead of σ_0] such that the function

$$H(s) = h_0(s) + \varepsilon_1 h_1(s) + \varepsilon_2 h_2(s) + \cdots = \sum C_n n^{-s}, \tag{10}$$

where

$$C_n = c_n^{(0)} + \varepsilon_1 c_n^{(1)} + \varepsilon_2 c_n^{(2)} + \cdots \tag{11}$$

for every sequence $\varepsilon_1, \varepsilon_2, \cdots$ such that

$$0 < \varepsilon_1 < e_1, \quad 0 < \varepsilon_2 < e_2, \cdots \tag{12}$$

will be regular in the half plane $\sigma > \sigma_0$ and in this half plane have the order of magnitude μ. (The series (11) converges and the two series in (10) are absolutely convergent for $\sigma \geqq \sigma_0 + \mu + 2$ with the same sum $H(s)$. This follows immediately from the previous remark since $e_N < E_N$ for $N = 1, 2, \cdots$.)

Proof. We have to prove that we can choose the positive numbers $e_N < E_N$ so that the function

$$H(s) = h_0(s) + \varepsilon_1 h_1(s) + \varepsilon_2 h_2(s) + \cdots$$

under the assumption (12) will be regular in the half plane $\sigma > \sigma_0$ and in this half plane satisfy the relation

$$H(s) = O(|t|^{\mu+\delta})$$

for every $\delta > 0$ but not for any $\delta < 0$. We divide the proof into two parts.

1°. In this part we choose the positive numbers $e_N < E_N$ so that the function $H(s)$ under the assumption (12) will be regular in the half plane $\sigma > \sigma_0$ and in this half plane equal to $O(|t|^{\mu+\delta})$ for every $\delta > 0$.

In the proof we shall use only that $\mu_N \leq \mu$ for $N = 0, 1, 2, \cdots$ and not that $\mu_N < \mu$ for $N = 1, 2, \cdots$. Let $\delta_1, \delta_2, \cdots$ be a sequence of positive numbers which tends to 0. On account of the assumptions there exist positive constants K_{mN} $(m = 1, 2, \cdots; N = 0, 1, 2, \cdots)$ such that

$$|h_N(s)| \leq K_{mN}(|t| + 1)^{\mu+\delta_m} \quad \text{for} \quad \sigma > \sigma_0.$$

We choose the constants $e_N < E_N$ so that

$$\sum_{N=m}^{\infty} e_N K_{mN}$$

is convergent for every $m = 1, 2, \cdots$. This may be done by subjecting the e_N to the following demands (only in a finite number for each e_N)

$$e_1 K_{11} < \frac{1}{2}, \quad e_2 K_{12} < \frac{1}{4}, \quad e_3 K_{13} < \frac{1}{8}, \cdots$$

$$e_2 K_{22} < \frac{1}{4}, \quad e_3 K_{23} < \frac{1}{8}, \cdots$$

$$e_3 K_{33} < \frac{1}{8}, \cdots$$

$$\cdots\cdots\cdots$$

Then we have under assumption (12)

$$|H(s)| \leq \left(|h_0(s)| + \varepsilon_1 |h_1(s)| + \cdots + \varepsilon_{m-1} |h_{m-1}(s)|\right) + \left(\varepsilon_m |h_m(s)| + \cdots\right) \leq$$

$$A_1(|t| + 1)^{\mu+\delta_m} + \left(\sum_{N=m}^{\infty} e_N K_{mN}\right)(|t| + 1)^{\mu+\delta_m} \leq$$

$$A_2(|t| + 1)^{\mu+\delta_m} \quad \text{for} \quad \sigma > \sigma_0,$$

where A_1 and A_2 are constants.

From this follows our above statement concerning the order of magnitude of $H(s)$. In order to see that $H(s)$ is regular for $\sigma > \sigma_0$ we remark that the series

$$h_0(s) + \varepsilon_1 h_1(s) + \varepsilon_2 h_2(s) + \cdots$$

in the half strip $\sigma > \sigma_0$, $|t| < T$, where T is any fixed positive number, will be majorized by the series

$$\left(K_{10} + \sum_{N=1}^{\infty} e_N K_{1N}\right)(T+1)^{\mu+\delta_1}$$

so that it is uniformly convergent in this half strip.

2°. In this part we choose the positive numbers $e_N < E_N$ so that $H(s)$ under the assumption (12) is not equal to $O(|t|^{\mu-\delta})$ in the half plane $\sigma > \sigma_0$ for any $\delta > 0$, or, in other words, that $H(s)$ is not equal to $o(|t|^{\mu-\delta})$ in the half plane $\sigma > \sigma_0$ for any $\delta > 0$. Thus it suffices to show that to every $\delta > 0$ there exist points $s = \sigma + it$ with $\sigma > \sigma_0$ and $|t|$ arbitrarily large such that

$$|H(s)| > \frac{1}{3}|t|^{\mu-\delta}.$$

We do this by showing that for a certain sequence of positive numbers $\delta_1, \delta_2, \cdots$ which tends to 0 there exists a corresponding sequence $s_1 = \sigma_1 + it_1$, $s_2 = \sigma_2 + it_2, \cdots$ with $\sigma_m > \sigma_0$ and $|t_m| \to \infty$ so that

$$|H(s_m)| > \frac{1}{3}|t_m|^{\mu-\delta_m} \quad \text{for} \quad \sigma > \sigma_0.$$

On account of the assumptions we know that to every $h_N(s)$, $N = 1, 2, \cdots$ there exists a positive number Δ_N such that

$$|h_N(s)| = o\left(|t|^{\mu-\Delta_N}\right) \quad \text{for} \quad \sigma > \sigma_0.$$

We now choose an arbitrary sequence of positive numbers $\delta_1, \delta_2, \cdots$ which tends to 0 and satisfies the conditions

$$\delta_2 \leq \Delta_1, \quad \delta_3 \leq \min(\Delta_1, \Delta_2), \cdots.$$

Our task is to choose the positive numbers $e_N < E_N$ in such a way that it is possible under the assumption (12) to find complex numbers s_m corresponding to the numbers δ_m with the above-mentioned properties. We shall do this in a sequence of steps.

First step. We choose a complex number $s_1 = \sigma_1 + it_1$ with $\sigma_1 > \sigma_0$, $|t_1| > 1$ so that

$$|h_0(s_1)| > |t_1|^{\mu - \delta_1}.$$

At the point $s = s_1$ the functions $h_1(s), h_2(s), \cdots$ assume certain values $k_{11}, k_{12}, \cdots$. We choose the positive constants $e_{11} < E_1$, $e_{12} < E_2, \cdots$ so that

$$\sum_{N=1}^{\infty} e_{1N} |k_{1N}|$$

is convergent with sum $< \frac{1}{2} |t_1|^{\mu_1 - \delta_1}$. Then for $0 < \varepsilon_N < e_{1N}$ $(N = 1, 2, \cdots)$ we have

$$|H(s_1)| \geqq |h_0(s_1)| - \sum_{N=1}^{\infty} e_{1N} |k_{1N}| > \frac{1}{2} |t_1|^{\mu - \delta_1}.$$

Second step. We choose $s_2 = \sigma_2 + it_2$ with $\sigma_2 > \sigma_0$, $|t_2| > 2$ so that

$$|h_0(s_2)| > |t_2|^{\mu - \delta_2}$$

and at the same time

$$E_1 |h_1(s_2)| < \frac{1}{3} |t_2|^{\mu - \delta_2}.$$

The latter inequality may be obtained since $\delta_2 \leqq \Delta_1$. At the point $s = s_2$ the functions $h_2(s), h_3(s), \cdots$ assume certain values $k_{22}, k_{23}, \cdots$. We choose the positive numbers $e_{22} < E_2$, $e_{23} < E_3, \cdots$ so that

$$\sum_{N=2}^{\infty} e_{2N} |k_{2N}|$$

is convergent with sum $< \frac{1}{3} |t_2|^{\mu - \delta_2}$. Then for $0 < \varepsilon_1 < E_1$, $0 < \varepsilon_N < e_{2N}$ $(N = 2, 3, \cdots)$ we have

$$|H(s_2)| \geqq |h_0(s_2)| - E_1 |h_1(s_2)| - \sum_{N=2}^{\infty} e_{2N} |k_{2N}| >$$

$$|t_2|^{\mu - \delta_2} - \frac{1}{3} |t_2|^{\mu - \delta_2} - \frac{1}{3} |t_2|^{\mu - \delta_2} = \frac{1}{3} |t_2|^{\mu - \delta_2}.$$

. . .

m^{th} *step.* We choose $s_m = \sigma_m + it_m$ with $\sigma_m > \sigma_0$ and $|t_m| > m$ so that

$$|h_0(s_m)| > |t_m|^{\mu - \delta_m}$$

and at the same time

$$E_1 \mid h_1(s_m) \mid + E_2 \mid h_2(s_m) \mid + \cdots + E_{m-1} \mid h_{m-1}(s_m) \mid < \frac{1}{3} \mid t_m \mid^{\mu - \delta_m}.$$

The latter inequality may be obtained since $\delta_m \leqq \min(\Delta_1, \cdots, \Delta_{m-1})$. At the point s_m the functions $h_m(s), h_{m+1}(s), \cdots$ assume certain values $k_{mm}, k_{m,m+1}, \cdots$. We choose positive constants $e_{mm} < E_m$, $e_{m,m+1} < E_{m+1}, \cdots$ so that

$$\sum_{N=m}^{\infty} e_{mN} \mid k_{mN} \mid$$

is convergent with sum $< \frac{1}{3} \mid t_m \mid^{\mu - \delta_m}$. Then for $0 < \varepsilon_1 < E_1, \cdots$, $0 < \varepsilon_{m-1} < E_{m-1}$, $0 < \varepsilon_N < e_{mN}$ $(N = m, m+1, \cdots)$ we have

$$\mid H(s_m) \mid > \mid h_0(s_m) \mid - (E_1 \mid h_1(s_m) \mid + \cdots + E_{m-1} \mid h_{m-1}(s_m) \mid) -$$
$$\sum_{N=m}^{\infty} e_{mN} \mid k_{mN} \mid > \mid t_m \mid^{\mu - \delta_m} - \frac{1}{3} \mid t_m \mid^{\mu - \delta_m} - \frac{1}{3} \mid t_m \mid^{\mu - \delta_m} = \frac{1}{3} \mid t_m \mid^{\mu - \delta_m}.$$

...

It appears from the above that the numbers

$$e_N = \min \{e_{1N}, \cdots, e_{NN}\} \quad (N = 1, 2, \cdots)$$

may be used in order to satisfy our demands under 2°.

Finally, for each N we choose the smaller one of the two numbers e_N found under 1° and 2° as our final e_N. These e_N satisfy the demands in Lemma 2. This completes the proof of Lemma 2.

§ 6. Proof of the Main Theorem.

We are now in a position to prove the Main Theorem stated in § 1. Since the function $f(s) = 0$ has $\omega_\mu = \omega_\psi = \Omega = -\infty$ we need only consider the following three cases: (α) $\omega_\mu = \omega_\psi = \Omega > -\infty$, ($\beta$) $\omega_\mu > \omega_\psi = \Omega > -\infty$, and the "general" case (γ) $\omega_\mu \geqq \omega_\psi > \Omega \geqq -\infty$.

As an example of the special case (α) we can obviously use the series

$$\zeta(s - \Omega + 1) = \sum n^{\Omega - 1} n^{-s}.$$

In fact, the series is absolutely convergent for $\sigma > \Omega$, and the function has a pole at $s = \Omega$.

The "intermediate" case (β) will be treated at the end of this section by specializing, and slightly modifying, the construction used in the "general" case (γ).

Let us therefore assume for the present that $\omega_\mu \geqq \omega_\psi > \Omega \geqq -\infty$. In the main, our Dirichlet series $\sum A_n n^{-s}$ is constructed by linear combination of infinitely many of the "bricks" from § 4, i. e. by linear combination of Dirichlet series whose summability function and order function have the form $\{\omega_1; a\}$ and $\{\omega_2; a\}$ with common $a \geqq 1$ and $0 \leqq \omega_2 - \omega_1 \leqq \frac{1}{a}$ (viz. with the vertical distance from the ψ-half-line to the μ-half-line $\geqq 0$ and $\leqq 1$). This construction, however, requires some caution because we have to build up at the same time two convex curves and because each of these curves may contain infinitely many vertices, i. e. points with different tangents from the right and the left.

We call a pair (T^ψ, T^μ) of parallel (perhaps coinciding) straight lines T^ψ and T^μ *a pair of supporting lines* (in a generalized sense) of our ψ-curve and our μ-curve when one of the lines T^ψ and T^μ is a proper supporting line of the corresponding curve at a point outside the real axis while the other line is defined by the upper position of all lines with the given slope which lie under the other curve. If the latter line contains at least one point of the curve in question, this line is of course a proper supporting line. In any case it is easily seen from the convexity of the two curves $\psi(\sigma)$ and $\mu(\sigma)$ and the relations $\psi(\sigma) \leqq \mu(\sigma) \leqq \psi(\sigma) + 1$ that the vertical distance from the line T^ψ to the line T^μ is $\geqq 0$ and $\leqq 1$. Furthermore, since $\psi'(\omega_\psi - 0) \leqq -1$ and $\mu'(\omega_\mu - 0) \leqq -1$ the slope $-a$ of the two lines is $\leqq -1$, i. e. $a \geqq 1$.

We start by choosing a denumerable set of abscissae $\sigma_1, \sigma_2, \cdots$ which lie everywhere dense in the interval $\Omega < \sigma < \omega_\mu$. These abscissae are chosen arbitrarily with the exception that we do not use any abscissa σ at which any of the functions $\psi(\sigma)$ and $\mu(\sigma)$ has different derivatives from the left and the right (i. e. which corresponds to a vertex on any of the two curves). For each of the above chosen abscissae σ_i which lie in the sub-interval $\Omega < \sigma < \omega_\psi$ of $\Omega < \sigma < \omega_\mu$ we consider both the supporting line S_i^ψ of the ψ-curve at the point $(\sigma_i, \psi(\sigma_i))$ and the supporting line S_i^μ of the μ-curve at the point $(\sigma_i, \mu(\sigma_i))$. For each of the abscissae

σ_i of the above chosen sequence which (if $\omega_\psi < \omega_\mu$) lie in the complementary sub-interval $\omega_\psi < \sigma < \omega_\mu$ of $\Omega < \sigma < \omega_\mu$ we consider only the supporting line S_i^μ of the μ-curve at the point σ_i. The supporting lines S_i^μ and (in the first case) S_i^ψ are uniquely determined since none of the two curves has a vertex at a point σ_i. For each of the abscissae σ_i which lie in the interval $\Omega < \sigma < \omega_\psi$ we now determine two pairs of supporting lines (T^ψ, T^μ) (which may coincide), one pair being determined by $T^\psi = S_i^\psi$, the other pair by $T^\mu = S_i^\mu$. For the first pair we mark the point $(\sigma_i, \psi(\sigma_i))$ on the line $T^\psi = S_i^\psi$; for the second pair we mark the point $(\sigma_i, \mu(\sigma_i))$ on the line $T^\mu = S_i^\mu$. For each of the abscissae σ_i which (if $\omega_\psi < \omega_\mu$) lie in the interval $\omega_\psi < \sigma < \omega_\mu$ we determine one pair of supporting lines (T^ψ, T^μ), namely the pair defined by $T^\mu = S_i^\mu$, and for this pair we mark the point $(\sigma_i, \mu(\sigma_i))$ on the line $T^\mu = S_i^\mu$. We arrange the pairs of supporting lines (T^ψ, T^μ) thus obtained (for each of our abscissae either one or two pairs) in a sequence

$$(T_1^\psi, T_1^\mu),\ (T_2^\psi, T_2^\mu), \cdots.$$

As mentioned above, we have marked for each of these pairs a point on one of its lines, T^ψ or T^μ. If we do not take notice of the marked points, it is evident that some of our pairs of supporting lines may coincide. (If for instance both the ψ-curve and the μ-curve are of the type $\{\omega; a\}$ with the same a, then all our pairs of supporting lines will be identical.) If such a coincidence between pairs occurs we shall only keep one of the coinciding pairs, but at the same time we shall change the point marking of the pairs according to the following specification. Let us assume that the pairs of supporting lines

$$(T_{m_1}^\psi, T_{m_1}^\mu), (T_{m_2}^\psi, T_{m_2}^\mu), \cdots$$

coincide.—For orientation we note that this sequence can either contain just two pairs of supporting lines, one with point-marking on the line T^ψ, the other with point-marking on the line T^μ, or the sequence will contain infinitely many pairs of supporting lines. This latter case will only occur when at least one of the curves $\psi(\sigma)$ or $\mu(\sigma)$ contains a straight segment outside the axis of abscissa.—As mentioned above, we keep only one of these pairs, but we now mark more points on the pair, namely all points on its T^{ψ}-line which are marked on one of the lines $T^{\psi}_{m_1}, T^{\psi}_{m_2}, \cdots$ as well as all points on its T^{μ}-line which are marked on one of the lines $T^{\mu}_{m_1}, T^{\mu}_{m_2}, \cdots$. If more than one point is marked on the line T^{ψ} we arrange these points in a sequence; analogously, if more than one point is marked on the line T^{μ} we arrange these points in a sequence.

The set of pairs of supporting lines (with their arranged marked points) obtained by the above procedure is now arranged in a (finite or infinite) sequence

$$\left(T^{\psi}_{\mathrm{I}}, T^{\mu}_{\mathrm{I}}\right), \left(T^{\psi}_{\mathrm{II}}, T^{\mu}_{\mathrm{II}}\right), \cdots, \left(T^{\psi}_{N}, T^{\mu}_{N}\right), \cdots.$$

It is plain that each of our abscissae σ_i which lie in the interval $\Omega < \sigma < \omega_{\psi}$ will occur as abscissa of a marked point on one of our lines T^{ψ} as well as on one of our lines T^{μ}, while each of the abscissae σ_i which (if $\omega_{\psi} < \omega_{\mu}$) lie in the interval $\omega_{\psi} < \sigma < \omega_{\mu}$ will occur as abscissa of a marked point on one of our lines T^{μ}.

For these pairs of supporting lines we introduce "bricks" in accordance with § 4, i. e. Dirichlet series

$$f_1(s) = \sum a_n^{(1)} n^{-s},\ f_2(s) = \sum a_n^{(2)} n^{-s}, \cdots,\ f_N(s) = \sum a_n^{(N)} n^{-s}, \cdots$$

such that those parts of the ψ-function and the μ-function of the series $f_N(s)$ where these functions are positive are determined by the half lines over the real axis which lie on T^{ψ}_N and T^{μ}_N, respectively. This is possible since the slope $-\alpha_N$ of the two lines is ≤ -1 and the vertical distance from T^{ψ}_N to T^{μ}_N is ≥ 0 and ≤ 1.

The series we are going to construct is formed by linear combination of these series $f_1(s), f_2(s), \cdots$; in fact, it has the form

$$F(s) = \varepsilon_1 f_1(s) + \varepsilon_2 f_2(s) + \cdots + \varepsilon_N f_N(s) + \cdots = \sum A_n n^{-s},$$

where $A_n = \sum_{N=1}^{\infty} \varepsilon_N a_n^{(N)}$. We shall show that we can choose the positive numbers $\varepsilon_1, \varepsilon_2, \cdots$ so that $\varepsilon_1 f_1(s) + \varepsilon_2 f_2(s) + \cdots$ is represented by a Dirichlet series $\sum A_n n^{-s}$ which for $\sigma > \Omega$ has its summability function $\Psi(\sigma)$ equal to the given function $\psi(\sigma)$ and its order function $M(\sigma)$ equal to the given function $\mu(\sigma)$. However, when $\Omega > -\infty$ we cannot always be sure that our construction yields a function $F(s)$ which does not have a limit abscissa of summability Ω_F smaller than the given number Ω.

In order to obtain the said properties of $F(s)$ it is enough to prove, *first, that the summability function* $\Psi(\sigma)$ *of* $F(s)$ *satisfies the equation* $\Psi(\sigma_i) = \psi(\sigma_i)$ *for those of our* σ_i *which lie in the interval* $\Omega < \sigma < \omega_\psi$ (this includes that Ω_F must be $\leq \Omega$) *and, secondly, that the order function* $M(\sigma)$ *of* $F(s)$ *satisfies the equation* $M(\sigma_i) = \mu(\sigma_i)$ *for all our* σ_i. In fact, the abscissae σ_i lie everywhere dense in the interval $\Omega < \sigma < \omega_\mu$; so for reasons of continuity we may conclude that the equations

$$\Psi(\sigma) = \psi(\sigma) \quad \text{and} \quad M(\sigma) = \mu(\sigma)$$

hold in the intervals $\Omega < \sigma \leq \omega_\psi$ and $\Omega < \sigma \leq \omega_\mu$, respectively; furthermore, since $\Psi(\omega_\psi) = \psi(\omega_\psi) = 0$ and $M(\omega_\mu) = \mu(\omega_\mu) = 0$, we get $\Psi(\sigma) = 0 = \psi(\sigma)$ for $\sigma > \omega_\psi$ and $M(\sigma) = 0 = \mu(\sigma)$ for $\sigma > \omega_\mu$ so that the above equations will hold in the whole interval $\Omega < \sigma < \infty$.

We remarked above that the constructed function $F(s)$ when $\Omega > -\infty$ might have $\Omega_F < \Omega$ and not $\Omega_F = \Omega$ as desired.

There are some cases with $\Omega > -\infty$ when automatically $\Omega_F = \Omega$, namely when $\psi'(\sigma - 0) \rightarrow -\infty$ or $\mu'(\sigma - 0) \rightarrow -\infty$ for $\sigma \rightarrow \Omega$. In fact, it is impossible in these cases to prolong the given ψ- and μ-curve to the left under preservation of their convexity, so that we can be sure that the constructed function $F(s)$ will have $\Omega_F = \Omega$ as desired.

In the other cases with $\Omega > -\infty$ we can prolong the ψ- and the μ-curve to the whole interval $-\infty < \sigma < \infty$ under preservation of all the properties demanded in the theorem, for instance by two parallel half lines with a common slope $\leq \min \left(\lim_{\sigma \rightarrow \Omega} \psi'(\sigma - 0), \lim_{\sigma \rightarrow \Omega} \mu'(\sigma - 0) \right)$. This we do before passing to the construction of $F(s)$, i. e. before choosing our σ_i.

The function $F(s)$ obtained will then be an entire function with these prolonged functions $\psi(\sigma)$ and $\mu(\sigma)$ as its summability function and order function, respectively. In order to obtain a function $F^*(s)$ from $F(s)$ which has the right Ω and without changing the Ψ-curve and the M-curve for $\sigma > \Omega$ we may for instance add the function

$$\zeta(s-\Omega+1) = \sum n^{\Omega-1} n^{-s}.$$

In this way we obtain a function $F^*(s)$ with all the desired properties.

We now pass to the actual construction of $F(s)$ referred to above. We determine the positive numbers $\varepsilon_1 = \varepsilon_1^*$, $\varepsilon_2 = \varepsilon_2^*, \cdots$ successively by the following procedure.

First step. We choose $\varepsilon_1 = \varepsilon_1^*$ as an arbitrary positive number. We consider the pair of supporting lines $\left(T_{\mathrm{I}}^{\psi}, T_{\mathrm{I}}^{\mu}\right)$ belonging to $f_1(s)$ with its marked points and distinguish between the following three cases.

1°. There exist marked points on the line T_{I}^{ψ}, but not on the line T_{I}^{μ}. If only one marked point is lying on T_{I}^{ψ} we denote its abscissa by σ_0 (where $\Omega < \sigma_0 < \omega_{\psi}$). If infinitely many marked points lie on T_{I}^{ψ} we denote by σ_0 (where $\Omega < \sigma_0 < \omega_{\psi}$) the abscissa of that point on T_{I}^{ψ} which comes first in the given ordering of the marked points on T_{I}^{ψ}. In the present case we are only interested in the Ψ-function at the point σ_0, and not in the M-function at this point.

We put the demand on the sequence $\varepsilon_2, \varepsilon_3, \cdots$ that

$$(1) \qquad F(s) = \varepsilon_1^* f_1(s) + \varepsilon_2 f_2(s) + \varepsilon_3 f_3(s) + \cdots = \sum A_n n^{-s}$$

is to have $\Psi(\sigma_0) = \psi(\sigma_0)$. In other words, we demand that the r_0^{th} abscissa of summability Λ_{r_0} of $F(s)$ where r_0 denotes the positive number $\psi(\sigma_0)$ is exactly equal to σ_0. We apply Lemma 1 of § 5 to the functions

$$g_0(s) = \varepsilon_1^* f_1(s),\ g_1(s) = f_2(s),\ g_2(s) = f_3(s), \cdots$$

and the numbers $\Lambda = \sigma_0$ and $r = r_0$ just determined. The supporting lines $T_{\mathrm{II}}^{\psi}, T_{\mathrm{III}}^{\psi}, \cdots$ of the ψ-curve cut the line $\sigma = \sigma_0$ below the point $(\sigma_0, \psi(\sigma_0))$ (because the point $(\sigma_0, \psi(\sigma_0))$ is no vertex on the ψ-curve). Hence the r_0^{th} abscissa of summability of the series $g_1(s), g_2(s), \cdots$ all lie to the left of σ_0 while the r_0^{th} abscissa of summability of $g_0(s)$ is equal to σ_0. It follows from Lemma 1 that there exist positive constants $e_{22}, e_{23}, \cdots$ with the property that the function (1) for $0 < \varepsilon_2 < e_{22}$, $0 < \varepsilon_3 < e_{23}, \cdots$ has its r_0^{th} abscissa of summability equal to σ_0, as desired.

2°. There exist marked points on the line T_{I}^{μ} but no marked points on the line T_{I}^{Ψ}. If only one marked point is lying on T_{I}^{μ} we denote its abscissa by σ_0 (where $\Omega < \sigma_0 < \omega_\mu$). If infinitely many marked point lie on T_{I}^{μ} we denote by σ_0 (where $\Omega < \sigma_0 < \omega_\mu$) the abscissa of that point on T_{I}^{μ} which comes first in the given ordering of the marked points on T_{I}^{μ}. In the present case we are only interested in the M-function at the point σ_0, and not in the Ψ-function at this point.

We put the demand on the sequence $\varepsilon_2, \varepsilon_3, \cdots$ that the function (1) must be regular for $\sigma > \sigma_0$ and have $M(\sigma_0) = \mu(\sigma_0)$. In other words, we demand that the function (1) is to be regular in the half plane $\sigma > \sigma_0$ and in this half plane have (exactly) the order of magnitude μ_0, where $\mu_0 = \mu(\sigma_0) > 0$. We apply Lemma 2 of § 5 to the functions

$$h_0(s) = \varepsilon_{\mathrm{I}}^{*} f_1(s),\ h_1(s) = f_2(s),\ h_2(s) = f_3(s), \cdots$$

and the numbers σ_0 and μ_0 just determined. The supporting lines $T_{\mathrm{II}}^{\mu}, T_{\mathrm{III}}^{\mu}, \cdots$ cut the line $\sigma = \sigma_0$ below the point $(\sigma_0, \mu(\sigma_0))$ (because the point $(\sigma_0, \mu(\sigma_0))$ is no vertex on the μ-curve). Hence the orders of magnitude of the functions $h_1(s), h_2(s), \cdots$ in the half plane $\sigma > \sigma_0$ are all $< \mu_0$, while the order of magnitude of the function $h_0(s)$ in the half plane $\sigma > \sigma_0$ is equal to μ_0. It follows from Lemma 2 that there exist positive constants $e_{22}, e_{23}, \cdots$ with the property that the function (1) for $0 < \varepsilon_2 < e_{22}, 0 < \varepsilon_3 < e_{23}, \cdots$ is regular in the half plane $\sigma > \sigma_0$ and has the order of magnitude μ_0 in this half plane, as desired.

3°. There exist marked points on the line T_{I}^{Ψ} as well as on the line T_{I}^{μ}. We consider two abscissae σ_0' and σ_0'' (they may coincide) where σ_0' denotes the abscissa of *the* marked point or the first of the marked points on the line T_{I}^{Ψ} while σ_0'' denotes the abscissa of *the* marked point or the first of the marked points on the line T_{I}^{μ}. By exactly the same considerations as under 1° and 2°, using the first time Lemma 1 and the second time Lemma 2, we find two sequences of positive numbers $e_{22}', e_{23}', \cdots$ and $e_{22}'', e_{23}'', \cdots$ such that the function (1) for $0 < \varepsilon_2 < e_{22}, 0 < \varepsilon_3 < e_{23}, \cdots$, where $e_{2j} = \min\ (e_{2j}', e_{2j}'')$ has $\Psi(\sigma_0') = \psi(\sigma_0')$, is regular for $\sigma > \sigma_0''$, and has $M(\sigma_0'') = \mu(\sigma_0'')$.

Summarizing, we have by this first step found a positive constant ε_1^* and positive constants $e_{22}, e_{23}, \cdots$ such that the function (1) for $0 < \varepsilon_2 < e_{22}$, $0 < \varepsilon_3 < e_{23}, \cdots$ has the property that its Ψ-curve will pass through *the* marked point or the first of the marked points on T_{I}^{ψ} (if such points exist) and the M-curve will pass through *the* marked point or the first of the marked points on T_{I}^{μ} (if such points exist).

...

N^{th} *step.* ($N \geqq 2$). We assume that by the $1^{\text{st}}, 2^{\text{nd}}, \cdots, (N-1)^{\text{th}}$ step we have determined positive constants $\varepsilon_1^*, \varepsilon_2^*, \cdots, \varepsilon_{N-1}^*$ and (by the $(N-1)^{\text{th}}$ step) positive constants $e_{N,j}$ $(j = N, N+1, \cdots)$ such that the function

$$F(s) = \varepsilon_1^* f_1(s) + \cdots + \varepsilon_{N-1}^* f_{N-1}(s) + \varepsilon_N f_N(s) + \\ + \varepsilon_{N+1} f_{N+1}(s) + \cdots = \sum A_n n^{-s} \tag{2}$$

for $0 < \varepsilon_N < e_{NN}$, $0 < \varepsilon_{N+1} < e_{N,N+1}, \cdots$ has the property that its Ψ-curve passes through the first $N-1$ of the marked points on T_{I}^{ψ}, through the first $N-2$ of the marked points on $T_{\mathrm{II}}^{\psi}, \cdots$, through the first of the marked points on T_{N-1}^{ψ}, and that its M-curve passes through the first $N-1$ of the marked points on T_{I}^{μ}, through the first $N-2$ of the marked points on $T_{\mathrm{II}}^{\mu}, \cdots$, through the first of the marked points on T_{N-1}^{μ}. It is plain how this is to be understood when one of the supporting lines T^{ψ} or T^{μ} only has one marked point or none at all.

We choose an arbitrary constant ε_N^* in the interval $0 < \varepsilon_N < e_{NN}$ and shall show that we can find positive constants $e_{N+1,N+1} \leqq e_{N,N+1}$, $e_{N+1,N+2} \leqq e_{N,N+2}, \cdots$ such that the function

$$F(s) = \varepsilon_1^* f_1(s) + \cdots + \varepsilon_N^* f_N(s) + \varepsilon_{N+1} f_{N+1}(s) + \\ + \varepsilon_{N+2} f_{N+2}(s) + \cdots = \sum A_n n^{-s} \tag{3}$$

for $0 < \varepsilon_{N+1} < e_{N+1,N+1}$, $0 < \varepsilon_{N+2} < e_{N+1,N+2}, \cdots$ has the property that its Ψ-curve passes through the first N of the marked points on T_{I}^{ψ}, through the first $N-1$ of the marked points on $T_{\mathrm{II}}^{\psi}, \cdots$, through the first of the marked points on T_N^{ψ}, and that its M-curve passes through the first N of the marked points on T_{I}^{μ}, through the first $N-1$ of the marked points on $T_{\mathrm{II}}^{\mu}, \cdots$, through the first of the marked points on T_N^{μ}.

It is evident that the conclusion from the $(N-1)^{\text{th}}$ step still holds good under the new conditions since

$$0 < \varepsilon_N^* < e_{NN} \quad \text{and} \quad 0 < \varepsilon_j < e_{N+1,j} \leq e_{N,j} \, (j = N+1, N+2, \cdots).$$

Thus we have only to make sure that the Ψ-curve (M-curve) passes through the N^{th} marked point on T_{I}^{ψ} (T_{I}^{μ}), through the $(N-1)^{\text{th}}$ marked point on T_{II}^{ψ} (T_{II}^{μ}), $\cdots$, through the first marked point on T_N^{ψ} (T_N^{μ}).

We consider the J^{th} pair of supporting lines (T_J^{ψ}, T_J^{μ}) $(J = \text{I}, \text{II}, \cdots, N)$. Let σ_0' (where $\Omega < \sigma_0' < \omega_\psi$) and σ_0'' (where $\Omega < \sigma_0'' < \omega_\mu$) denote the abscissae of the $(N+1-J)^{\text{th}}$ marked point on the lines T_J^{ψ} and T_J^{μ}, respectively (if they exist).

First, we put the demand on the sequence $\varepsilon_{N+1}, \varepsilon_{N+2}, \cdots$ that the function (3) (if σ_0' exists) has $\Psi(\sigma_0') = \psi(\sigma_0')$. In other words, we demand that the r_0^{th} abscissa of summability A_{r_0} of $F(s)$, where r_0 denotes the positive number $\psi(\sigma_0')$, is exactly equal to σ_0'. We apply Lemma 1 of § 5 to the functions

$$g_0(s) = \varepsilon_{\text{I}}^* f_1(s) + \cdots + \varepsilon_J^* f_J(s) + \cdots + \varepsilon_N^* f_N(s),$$

$$g_1(s) = f_{N+1}(s), \; g_2(s) = f_{N+2}(s), \cdots$$

and the numbers $A = \sigma_0'$ and $r = r_0$ just determined. The supporting lines T_P^{ψ} $(P \neq J)$ of the ψ-curve cut the line $\sigma = \sigma_0'$ below the point $(\sigma_0', \psi(\sigma_0'))$ (because the point $(\sigma_0', \psi(\sigma_0'))$ is no vertex on the ψ-curve). Hence the r_0^{th} abscissae of summability of the series $f_P(s)$, $P \neq J$, all lie to the left of σ_0', while the r_0^{th} abscissa of summability of $f_J(s)$ is equal to σ_0'. It follows immediately that the r_0^{th} abscissa of summability of $g_0(s)$ is equal to σ_0', while the r_0^{th} abscissae of summability of $g_1(s)$, $g_2(s)$, $\cdots$ are smaller than σ_0'. It follows from Lemma 1 that there exist positive constants ${}^{J}e_{N+1,N+1}'$, ${}^{J}e_{N+1,N+2}'$, $\cdots$ with the property that the function (3) for $0 < \varepsilon_{N+1} < {}^{J}e_{N+1,N+1}'$, $0 < \varepsilon_{N+2} < {}^{J}e_{N+1,N+2}'$, $\cdots$ has its r_0^{th} abscissa of summability A_{r_0} equal to σ_0'.

Next, we put the demand on the sequence $\varepsilon_{N+1}, \varepsilon_{N+2}, \cdots$ that the function (3) (if σ_0'' exists) must be regular for $\sigma > \sigma_0''$ and have $M(\sigma_0'') = \mu(\sigma_0'')$. In other words, we demand that the function (3) is to be regular in the half plane $\sigma > \sigma_0''$ and in this half plane have (exactly) the order of magnitude μ_0 where $\mu_0 = \mu(\sigma_0'') > 0$. We apply Lemma 2 of § 5 to the functions

$$h_0(s) = \varepsilon_1^* f_1(s) + \cdots + \varepsilon_J^* f_J(s) + \cdots + \varepsilon_N^* f_N(s),$$

$$h_1(s) = f_{N+1}(s),\ h_2(s) = f_{N+2}(s), \cdots$$

and the numbers σ_0'' and μ_0 just determined. The supporting lines T_P^μ, $P \neq J$, cut the line $\sigma = \sigma_0''$ below the point $(\sigma_0'', \mu(\sigma_0''))$ (because the point $(\sigma_0'', \mu(\sigma_0''))$ is no vertex on the μ-curve). Hence the orders of magnitude of the functions $f_P(s)$ in the half plane $\sigma > \sigma_0''$ for $P \neq J$ are smaller than μ_0, while the function $f_J(s)$ in the half plane $\sigma > \sigma_0''$ has the order of magnitude μ_0. It follows immediately that the order of magnitude of the function $h_0(s)$ in the half plane $\sigma > \sigma_0''$ is equal to μ_0, while the orders of magnitude of the functions $h_1(s), h_2(s), \cdots$ in the half plane $\sigma > \sigma_0''$ are smaller than μ_0. It follows from Lemma 2 that there exist positive constants ${}^{J}e''_{N+1,N+1}$, ${}^{J}e''_{N+1,N+2}, \cdots$ with the property that the function (3) for $0 < \varepsilon_{N+1} < {}^{J}e''_{N+1,N+1}$, $0 < \varepsilon_{N+2} < {}^{J}e''_{N+1,N+2}, \cdots$ is regular for $\sigma > \sigma_0''$ and has $M(\sigma_0'') = \mu(\sigma_0'')$.

It follows from the above that the numbers

$$e_{N+1,j} = \min\left\{e_{N,j};\ {}^{1}e'_{N+1,j}, \cdots, {}^{N}e'_{N+1,j};\ {}^{1}e''_{N+1,j}, \cdots, {}^{N}e''_{N+1,j}\right\}$$
$$(j = N+1, N+2, \cdots)$$

have the desired properties (under step N).

. . .

The conclusion is still missing, namely that the sequence $\varepsilon_1^*, \varepsilon_2^*, \cdots$ found above is such that the function

$$F(s) = \varepsilon_1^* f_1(s) + \varepsilon_2^* f_2(s) + \cdots = \sum A_n n^{-s} \tag{4}$$

has the desired properties. This, however, follows at once from the remark that

$$0 < \varepsilon_{N+1}^* < e_{N+1,N+1} \leqq e_{N,N+1}$$

$$0 < \varepsilon_{N+2}^* < e_{N+2,N+2} \leqq e_{N+1,N+2} \leqq e_{N,N+2}$$

. . . .

so that (4) gets the properties of (3) from the arbitrary step N $(N = 1, 2, \cdots)$, q. e. d.

This completes the proof of the Main Theorem in the "general" case (γ) $\omega_\mu \geqq \omega_\psi > \Omega \geqq -\infty$.

The remaining case $(\beta)\ \omega_\mu > \omega_\psi = \Omega > -\infty$ can be treated in a similar way as the general case. However, a small modification is necessary, due to the fact that the ψ-curve does not leave the real axis, but consists of the interval $\Omega < \sigma < \infty$ on the real axis. If we are in a case where the pair of functions $\psi(\sigma)$ and $\mu(\sigma)$ can be prolonged no modification is of course necessary since the prolonged curves fall under case (γ). In any case, the "bricks" $f_1(s), f_2(s) \cdots$ are obtained in the same way as before, but if we proceed as before (in the case where $\psi(\sigma)$ and $\mu(\sigma)$ could not be prolonged) by the determination of the numbers $\varepsilon_1^*, \varepsilon_2^*, \cdots$ it is plain, since no marked points occur on the lines T^ψ of our pairs of supporting lines $\left(T_N^\psi, T_N^\mu\right)$, that we have taken care only of the M-function, but not of the Ψ-function. However, from the determination of the pairs $\left(T_N^\psi, T_N^\mu\right)$ it follows that all the Dirichlet series $f_N(s)$ are convergent for $\sigma > \Omega$, for all the lines T_N^ψ pass through the end-point Ω of the ψ-curve.

In order to obtain that (4) also becomes convergent for $\sigma > \Omega$, and hence $\Psi(\sigma) = 0$ for $\sigma > \Omega$ as desired, we choose a sequence $\sigma_1^* > \sigma_2^* > \cdots \rightarrow \Omega$. By our first step we add the demand to the previous demands that (1) is also to be convergent for $s = \sigma_1^*$, and in order to obtain this situation we use a result obtained in the first part of the proof of Lemma 1 in the case $r = 0$, namely the result that if the Dirichlet series $g_1(s), g_2(s), \cdots$ are summable of the r^{th} order at the point $s = A$, then the positive numbers $e_1, e_2, \cdots$ can be chosen so that the Dirichlet series $G^*(s) = \varepsilon_1 g_1(s) + \varepsilon_2 g_2(s) + \cdots = \sum B_n^* n^{-s}$ becomes summable of the r^{th} order at the point $s = A$ when only $0 < \varepsilon_1 < e_1$, $0 < \varepsilon_2 < e_2, \cdots$. In our N^{th} step we add the demand to the previous demands that (3) is also to be convergent for $s = \sigma_N^*$. Except for this slight modification our previous method remains unchanged.

Thus the proof of our Main Theorem is completed.

References.

[1] A. F. ANDERSEN. Studier over Cesàro's Summabilitetsmetode. Doctoral Dissertation, Copenhagen 1921.

[2] H. BOHR. Ueber die Summabilität Dirichletscher Reihen. Nachr. Ges. Wiss. Göttingen. Math. Phys. Kl. 1909, 247-262.

[3] H. BOHR. Bidrag til de Dirichlet'ske Rækkers Theori. Doctoral Dissertation, Copenhagen 1910.

[4] H. BOHR. On the Convergence Problem for Dirichlet Series. Dan. Mat. Fys. Medd. **25**, no. 6 (1949).

[5] H. BOHR. Zur Theorie der Dirichletschen Reihen. Math. Z. 52 (1950), 709-722.

[6] G. H. HARDY and M. RIESZ. The General Theory of Dirichlet Series. Cambridge Tracts, no. 18, Cambridge 1915.

[7] M. RIESZ. Sur les séries de Dirichlet. C. R. Acad. Sci. Paris 148 (1909), 1658-1660.

[8] M. RIESZ. Sur un théorème de la moyenne et ses applications. Acta Sci. Math. Szeged 1 (1923), 114-126.

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